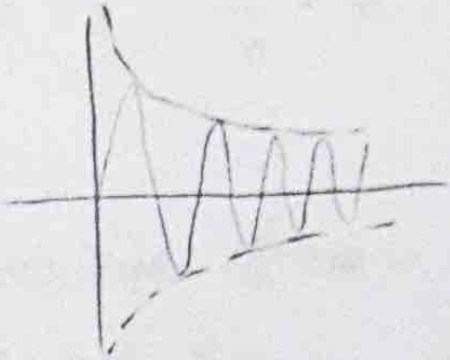
Exercise 2.2.1.4



a. $x_n = \frac{\sin 2n}{n}, \quad \varepsilon = 0.1$

x_n is messy, and we can get a much nicer bound.

$$\left|\frac{\sin 2n}{n}\right| \leq \frac{1}{n} \quad \text{since} \quad \sin 2n \in [-1, 1]$$

Now, we can prove that $\frac{1}{n} \to 0$, meaning that:

$$(\forall \varepsilon > 0)(\exists N)(n \geq N)\left[\left|\frac{1}{n}\right| < \varepsilon\right]$$

We set out to show that an algorithm for finding N given ε exists

$$\left|\frac{1}{n}\right| < \varepsilon$$

is equivalent to

$$\frac{1}{n} < \varepsilon \quad \text{since} \quad n \in \mathbb{N}$$

$$\text{so} \quad n > \frac{1}{\varepsilon}$$

so

$$(\forall \varepsilon > 0)(\forall N > \tfrac{1}{3})(\forall n \geq N)\left[\left|\frac{1}{n}\right| < \varepsilon\right]$$

$\frac{1}{n}$ is thus a convergent sequence with limit 0.

since $x_n \leq \frac{1}{n}$, $(\forall n \geq N)[|x_n| < \varepsilon]$

so if $\varepsilon = 0.1$, $N > \frac{1}{0.1} = 10$

$\therefore$ we can choose $N = 11$

$4n^3 + 3n$ $\quad n-6$

b. $x_n = \frac{2^n}{n!} \qquad \varepsilon = 0,001.$

$$x_{n+1} = x_n \cdot \frac{2}{n}$$

so $x_{n+1} < x_n \ (\forall n \geqslant 2)$

so if $x_n = 0,001$, we know that, we say $N := n$, such that

$$(\varepsilon = 0,001)(N \text{ as above})(\forall n \geqslant N)\,[x_n < 0,001]$$

on such x_n is $\frac{2^{10}}{10!}$, so $N = 10$ is valid.

⑦ $x_n = \dfrac{4n^3 + 3n}{n^3 - 6}$ $\varepsilon = 10^{-6}$

$$x_n = \frac{4 + \frac{3}{n^2}}{1 - \frac{6}{n^3}}$$, so we can guess that $x_n \to 4$

or that

$$(\forall \varepsilon > 0)(\exists N)(\forall n \geq N)\left[\left|\frac{4n^3 + 3n}{n^3 - 6} - 4\right| < \varepsilon\right]$$

so can we find an algorithm to generate an N for an a ε?

$$\left|\frac{4n^3 + 3n}{n^3 - 6} - 4\right| < \varepsilon$$

$$\left|\frac{4n^3 + 3n - 4n^3 + 24}{n^3 - 6}\right| < \varepsilon$$

$$\left|\frac{3n + 24}{n^3 - 6}\right| < \varepsilon$$

$$\frac{3n + 24}{|n^3 - 6|} < \varepsilon$$

let's find an upper bound for $\dfrac{3n + 24}{|n^3 - 6|}$

$$4n \geq 3n + 24$$

$$n \geq 24$$

and

$$n^3 \geq |n^3 - 6|$$

for $n^3 \geq -(n^3 - 6)$

$$2n^3 \geq 6$$

$$n \geq \sqrt{3}$$

Then we can find an N for which

$\frac{4}{n^2} < \varepsilon$, since if $\frac{4}{n^2} < \varepsilon$,

so will x_n.

$$\frac{2}{\sqrt{\varepsilon}} < n$$

for $\varepsilon = 10^{-6}$

$$n > \frac{2}{\sqrt{10^{-6}}}$$

$$n > 2000$$

so we can choose $N = 20001$